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A Sequence of Linear Polynomial Operators and Their Approximation-Theoretic Properties*

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0. INTRODUCTION

Let

$$E_n = \{x_1^{(n)},..., x_n^{(n)}\}, \qquad -1 \leqslant x_1^{(n)} < \cdots < x_n^{(n)} \leqslant 1,$$

denote the n-th row of a triangular matrix E and let $f(x)$ be defined in $[-1, 1]$. The polynomial $L_n(f, E_n) \equiv L_{n0}(f; x)$ of degree $n - 1$ interpolating f on E_n has been, since Newton and Lagrange, the subject of many investigations. It is a well-known result [9, p. 5] of Faber and Bernstein that

(1) for every matrix E, there exists a continuous function $f(x)$ on $[-1, 1]$ for which the sequence $\{L_{n0}(f; x)\}$ does not converge uniformly.

However, Fejér [6] has shown that

(2) if the Lebesgue constant $\lambda_n(E) < cn^\beta$, $0 < \beta < 1$, then the polynomials $L_{n0}(f; x)$ converge to $f(x)$, uniformly in $[-1, 1]$, if $f \in \text{Lip}\, \gamma$, $\gamma > \beta$.

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On the other hand, Erdös and Turán [5] have shown that

(3) if the points of E_n are the zeros of a polynomial Q_n of degree n, where $\{Q_n\}$ is an orthogonal sequence with respect to a weight function $w(x) \geqslant M > 0$, $-1 \leqslant x \leqslant 1$, then $\{L_{n0}(f; x)\}$ converges in the mean square to $f(x)$, even when $f(x)$ is only R-integrable.

Later Erdös and Feldheim [4] pointed out that

(4) for the zeros of the Tchebycheff polynomial of the first kind an even stronger result holds:

$$\lim_{n\to\infty} \int_{-1}^{1} | L_{n0}(f; x) - f(x)|^p (1 - x^2)^{-1/2}\, dx = 0, \qquad p = 1, 2,...$$

while for the zeros of the Tchebycheff polynomials $U_n(x)$ of the second kind $(U_n(x) = \sin(n + 1)\theta/\sin \theta, x = \cos \theta)$ there exists a continuous function $f(x)$ for which $\int_{-1}^{1} (L_{n0}(f; x) - f(x))^2\, dx$ approaches infinity as n increases.

For other related results see Feldheim [8]. More recently Askey [1] has shown that if E_n consists of the zeros of $Q_n^{(\alpha+1/2)}(x)$, the ultraspherical polynomial, $\alpha \geqslant -1/2$, then for every continuous f

$$\lim_{n\to\infty} \int_{-1}^{1} | L_{n0}(f; x) - f(x)|^p (1 - x^2)^{\alpha}\, dx = 0 \tag{0.1}$$

if $p < 4(\alpha + 1)/(2\alpha + 1)$, while if $p \geqslant 4(\alpha + 1)/(2\alpha + 1)$ there exists a continuous function $f(x)$ for which (0.1) fails. In the complex domain, Walsh and Sharma [16] proved

(5) the mean square convergence of $L_{n0}(f; z)$ to $f(x)$ on the unit circle, when E_n consists of the n-th roots of unity and $f(x)$ is analytic in $| z | < 1$ and continuous in $| z | \leqslant 1$.

The object of this paper is to give a scheme for defining a linear polynomial operator $L_{nr}(f; x)$ for any given integer r, $0 \leqslant r \leqslant n - 1$, which reduces for $r = 0$ to the Lagrange interpolation polynomial and which for $r = 1$ gives the so-called next-to-interpolatory polynomial (cf. Motzkin and Sharma [10]). We show that for fixed r, these polynomials share many of the convergence properties of the Lagrange polynomials including statements (1)–(5). We first develop (in Section 1) a general matrix-theoretic rank-diminishing procedure, a special case of which yields the polynomial operators L_{nr}.

1. Preliminaries on Matrices

1.1. In Section 1, we denote matrices by italic capitals, square matrices by greek capitals, the rank of A by $A^{\cdot}$, the transpose by A^T, rows by b or b'; j means a row consisting of zeros and one 1, as well as the position number of that 1; correspondingly we use c and k for columns. Then jA, Ak, jAk are a row, a column and an element of A; $A \setminus Ak$ means A with Ak deleted.

1.2. If the columns of A depend on some of their linear combinations: $A = AC \cdot D$, then the columns of BA depend on the corresponding linear combinations: $BA = BAC \cdot D$. But if, for some column c of C, $BAc = 0$ then the columns of BA depend already on $BA(C \setminus c)$.

1.3. *If* $Ac \neq 0$, $BAc = 0$ *then* $(BA)^{\cdot} < A^{\cdot}$. One proof uses 1.2 and the fact that there exists C with $A^{\cdot}$ columns one of which is c such that $(AC)^{\cdot} = A^{\cdot}$.

1.4. Lemma. *If* $\Gamma = bAc - Acb$ *then* (1) $\Gamma Ac = 0$, (2) $(\Gamma A)^{\cdot} < A^{\cdot}$ *if* $A \neq 0$, (3) $b\Gamma = 0$, (4) $b'A = 0$ *implies* $b'\Gamma A = 0$, (5) $\Gamma A = A\Delta$ *where* $\Delta = bAc - cbA$.

Proof. (1) follows from $j\Gamma Ac = bAc \cdot jAc - jAcbAc = 0$. (2) follows from (1) and 1.3 if $Ac \neq 0$; if $Ac = 0$ then $\Gamma = 0$, $\Gamma A = 0$. We have (3) by $b\Gamma = bAc \cdot b - bAcb$, (4) by $b'\Gamma A = bAc \cdot b'A - b'AcbA$, (5) by $bAc \cdot A = A \cdot bAc$, where bAc denotes two scalar matrices of possibly different sizes. (In general, $\phi = \lambda - AF$, $\psi = \lambda - FA$, with scalar λ, implies $\phi A = A\psi$.) Note that for $bAc \neq 0$, $\Gamma' = \Gamma/bAc = 1 - Acb/bAc$ and $\Delta' = \Delta/bAc$ have the same properties.

1.5. By assertion (2) of the lemma a general *rank diminishing algorithm* can be defined as follows. Choose b and c and replace A by

$$\Gamma A = A\Delta = bAc \cdot A - AcbA.$$

Now choose new b and c and continue. Then 0 is reached after at most $A^{\cdot}$ steps. But, by (3) and (4), if any b is used again at the next or some later step, 0 is reached at that step. By (5), the same holds for the reuse of c.

The variant $\Gamma'A$ has the same properties but halts when $bAc = 0$.

1.6. If the columns of 1 are consecutively chosen as c, then by (1) of the lemma, the first columns in the resulting matrices in turn become and stay 0 and may as well be omitted. This amounts to replacing, at each step, A by $A(\Delta \setminus \Delta k)$ or $A(\Delta' \setminus \Delta' k)$, where k is the first column of 1. We have:

If the columns of A are independent, so are those of

$$A(\Delta' \setminus \Delta' k), \qquad \Delta' = 1 - kbA/bAk.$$

Proof. Independence of the columns of A can be written $BA = 1$. Denoting the first row of 1 by j there follows $(B \setminus jB)\, A(\Delta' \setminus \Delta'k) = (1 \setminus j)(\Delta' \setminus \Delta'k) = 1$.

As the number of columns of A decreases, B loses its first rows.

1.7. For a real A, choose $c = k$ as in 1.6 and b such that $jb^T = \operatorname{sgn} jAk$ for all j; here sgn 0 is arbitrary subject to $-1 \leqslant \operatorname{sgn} 0 \leqslant 1$. In this case, if one starts with an invertible A, at the first step the signs of the highest order determinants are the same as those for the remaining rows of A^{-1}.

2. The Polynomial Algorithm

To the matrix A in Section 1 there correspond the polynomials $A\xi$; $\xi = (..., x^2, x, 1)^T$ and for given distinct $x_1, x_2, ...,$ the polynomial operator

$$fA = (f(x_1), f(x_2), ...)\, A,$$

which assigns to every function f the polynomial $fA\xi$. In particular, for Λ defined by

$$j\Lambda\xi = \prod_{i \neq j} \frac{x - x_i}{x_j - x_i},$$

$f\Lambda\xi$ is the interpolating polynomial to f; Λ^{-1} is the Vandermondian of $x_1, x_2, ...$. In fact if we denote by s_m the m-th elementary symmetric function in the n variables $x_1, x_2, ..., x_n$:

$$s_m = s_m(x_1, ..., x_n) = \sum x_{\nu_1} x_{\nu_2} \cdots x_{\nu_m}, \qquad 1 \leqslant m \leqslant n, \quad s_0 = 1 \quad (2.1)$$

and by $s_m^{(\nu)}$ the m-th elementary symmetric function in the $n - 1$ variables with x_ν missing; i.e.,

$$s_m^{(\nu)} = s_m(x_1, ..., x_{\nu-1}, x_{\nu+1}, .., x_n), \qquad (2.2)$$

then we have $\Lambda = (\lambda_{jk})$, where

$$\lambda_{jk} = (-1)^{k-1} \left[s_{n-k}^{(j)} \Big/ \prod_{\nu \neq j} (x_\nu - x_j) \right].$$

Let $E = \{z_1, ..., z_n\}$ be a set of n (distinct) points in the complex plane and let $l_{j0}(z)$ $(j = 1, ..., n)$ be the fundamental polynomials of degree $n - 1$ of Lagrange interpolation defined by

$$l_{j0}(z_k) = \delta_{jk}, \qquad j = 1, ..., n. \qquad (2.3)$$

The Lagrange interpolation operator $f\Lambda \equiv L_{n0}(f; z)$ is then given by

$$L_{n0}(f; z) = \sum_1^n f_j l_{j0}, \qquad \text{where} \quad f_j = f(z_j). \qquad (2.4)$$

If we set

$$\omega(z) = \prod_1^n (z - z_j), \qquad \omega_j = (\omega'(z_j))^{-1} = \prod_{k \neq j} (z_j - z_k)^{-1} \neq 0,$$

then

$$l_{j0}(z) = \omega_j \frac{\omega(z)}{z - z_j} = \omega_j(z^{n-1} - s_1^{(j)} z^{n-2} + s_2^{(j)} z^{n-3} - \cdots) \tag{2.5}$$

where $s_m^{(j)}$ is given by (2.2). Denote by l_{j0}^* the coefficient of z^{n-1} in l_{j0} ; then $l_{j0}^* = \omega_j \neq 0$. If $w_1 ,..., w_n$ are given positive numbers, set

$$\lambda_0(z) = \sum_1^n w_j^{-1} l_{j0}(z)/\text{sgn}\, l_{j0}^* = \sum_1^{n-1} (-1)^k \sum_1^n w_j^{-1} \mid \omega_j \mid s_k^{(j)} z^{n-k-1}. \tag{2.6}$$

Then the coefficient of z^{n-1} in $\lambda_0(z)$ is given by

$$\lambda_0{}^* = \sum_1^n \mid l_{j0}^* \mid w_j^{-1}. \tag{2.7}$$

If we now form the polynomials $l_{j1}(z)$ of degree $n - 2$ given by

$$l_{j1}(z) = l_{j0}(z) - (l_{j0}^*/\lambda_0{}^*)\, \lambda_0(z), \qquad j = 1,..., n, \tag{2.8}$$

then we can define the polynomial operator $L_{n1}(f; z)$ as follows:

$$L_{n1}(f; z) = \sum_1^n f_j l_{j1}(z). \tag{2.9}$$

This process of determining the polynomials l_{j1} from the polynomials l_{j0} can be iterated r times. For simplicity, from here on let $w_1 = \cdots = w_n = 1$. If r is a fixed integer, $1 \leqslant r \leqslant n - 1$, suppose we have already formed the polynomials $\{l_{j,r-1}\}_1^n$ of degree $n - r$. If $l_{j,r-1}^*$ and λ_{r-1}^* denote the coefficient of z^{n-r} in $l_{j,r-1}$ and λ_{r-1} , we set

$$l_{jr}(z) = l_{j,r-1}(z) - (l_{j,r-1}^*/\lambda_{r-1}^*)\, \lambda_{r-1}(z), \tag{2.10}$$

$$\lambda_{r-1}(z) = \sum_{j \in I_1} l_{j,r-1}(\text{sgn}\, l_{j,r-1}^*)^{-1} + \sum_{j \in I_2} \epsilon_j l_{j,r-1} , \qquad \mid \epsilon_j \mid = 1, \tag{2.11}$$

where $I_1 = \{j \mid l_{j,r-1}^* \neq 0\}$ and $I_2 = \{j \mid l_{j,r-1}^* = 0\}$. The linear polynomial operator

$$L_{nr}(f; z) = \sum_1^n f_j l_{jr} \tag{2.12}$$

maps functions into polynomials of degree $\leqslant n - r - 1$. The possible presence of arbitrary ϵ_j's, $|\epsilon_j| = 1$, brings in an indeterminacy in the algorithm (2.10) which we discuss in some detail in Section 3. However, we still have

LEMMA 1. *The linear operators* $L_{nr}(f; z)$ *given by* (2.1) *are projection operators onto the space of polynomials of degree* $\leqslant n - r - 1$. *Also*

$$\sum_1^n l_{jr}^* z_j^m = 0, \qquad m = 0, 1,..., n - r - 2, \qquad (2.13)$$
$$= 1, \qquad m = n - r - 1.$$

In particular, we have

$$\sum_1^n l_{j,n-1}^* = 1. \qquad (2.14)$$

Proof. For $r = 0$, the lemma is well known as a reproducing property of Lagrange interpolation which is exact for polynomials of degree $\leqslant n - 1$. This gives (2.13) for $r = 0$. The proof is now completed by induction on r, using (2.13) and (2.10).

The above lemma is independent of the arbitrary ϵ_j's, $|\epsilon_j| = 1$, which occurs in l_{kr} when $l_{j,r-1}^*$ vanishes. Formula (2.13) guarantees that the $l_{j,r-1}^*$ $(j = 0, 1,..., n - 1)$ can not all vanish. We now obtain an upper bound on $|L_{nr}(f; z)|$ independent of all ϵ_j's that may occur in (2.12). We have

LEMMA 2. *If* $max_i |f_i| = M$, *then for any given* r, $1 \leqslant r \leqslant n - 1$, *we have*

$$|L_{nr}(f; z)| \leqslant 2^r M \sum_1^n |l_{k0}(z)|. \qquad (2.15)$$

Proof. Denote by I_{r-1} the set of indices for which $l_{j,r-1}^* \neq 0$ and by J_{r-1} the complementary set. Then using (2.10) we have

$$\lambda_{r-1}^* L_{nr}(f; z) = \sum_1^n f_i (l_{i,r-1} \lambda_{r-1}^* - l_{i,r-1}^* \lambda_{r-1})$$
$$= \sum_{i=1}^n f_i \Big\{ l_{i,r-1} \sum_{j \in I_{r-1}} |l_{j,r-1}^*| - l_{i,r-1}^* \sum_{j \in I_{r-1}} l_{j,r-1} (\operatorname{sgn} l_{j,r-1}^*)^{-1}$$
$$- l_{i,r-1}^* \sum_{j \in J_{r-1}} \epsilon_j l_{j,r-1} \Big\}$$
$$= \sum_{j \in I_{r-1}} l_{j,r-1} \alpha_{j,1}(f) + \sum_{j \in I_{r-1}} l_{j,r-1} \beta_{j,1}(f),$$

where

$$\alpha_{j,1}(f) = \sum_{i \in I_{r-1}} l_{i,r-1}^* \{f_i(\operatorname{sgn} l_{i,r-1}^*)^{-1} - f_i(\operatorname{sgn} l_{j,r-1}^*)^{-1}\}, \qquad j \in I_{r-1},$$

$$\beta_{j,1}(f) = \sum_{i \in I_{r-1}} l_{i,r-1}^* \{f_j(\operatorname{sgn} l_{i,r-1}^*)^{-1} - f_i \epsilon_j\}, \qquad j \in J_{r-1}. \tag{2.16}$$

Then

$$\max\{|\alpha_{j,1}(f)|, |\beta_{j,1}(f)|\} \leqslant 2M \sum_{k \in I_{r-1}} |l_{k,r-1}^*| = 2M\lambda_{r-1}^*.$$

If $I_{r-2} = \{i \mid l_{i,r-2}^* \neq 0\}$, $J_{r-2} = \{j \mid l_{j,r-2}^* = 0\}$ and if $f^{(1)}$ denotes a function such that

$$f_j^{(1)} \equiv f^{(1)}(z_j) = \alpha_{j,1}(f), \qquad j \in I_{r-1},$$
$$= \beta_{j,1}(f), \qquad j \in J_{r-1},$$

then it is easy to see that

$$\lambda_{r-1}^* \lambda_{r-2}^* L_{nr}(f; z) = \sum_{j \in I_{r-2}} l_{j,r-2} \alpha_{j,2}(f^{(1)}) + \sum_{j \in J_{r-2}} l_{j,r-2} \beta_{j,2}(f^{(1)}),$$

where $\alpha_{j,2}(f^{(1)})$, $\beta_{j,2}(f^{(1)})$ are defined in a way analogous to (2.16). Also

$$\max(|\alpha_{j,2}(f^{(1)}), |\beta_{j,2}(f^{(1)})|) \leqslant 2^2 M \lambda_{r-1}^* \lambda_{r-2}^*.$$

Repeating the above process r times, we finally have

$$\prod_0^{r-1} \lambda_k^* L_{nr}(f; z) = \sum_1^n \alpha_{k,r} l_{k0} \tag{2.17}$$

where $|\alpha_{k,r}| \leqslant 2^r M(\prod_0^{r-1} \lambda_k^*)$. Taking absolute values in (2.17) completes the proof.

3. Special Sets E and Indeterminacy

In general, it is very difficult to compute the numbers $l_{j,r-1}^*$ and λ_{r-1}^*. For special E, it might also happen that for some r and j, $l_{j,r-1}^* = 0$ but from (2.13) it is clear that $l_{j,r-1}^*$ can not vanish for all j and hence λ_{r-1}^* can never be zero. However, as explained in Section 2, the vanishing of $l_{j,r-1}^*$ brings in an indeterminacy in the linear operators.

If l_{k0}^{**}, λ_0^{**} denote the coefficient of z^{n-2} in l_{k0} and λ_0, respectively, then $l_{k1}^* = 0$ is equivalent to $\lambda_0^* l_{k0}^{**} = \lambda_0^{**} l_{k0}^*$, namely, to $\lambda_0^*(x_k - \sum x_j) = \lambda_0^{**}$, i.e., $x_k = (\lambda_0^{**}/\lambda_0^*) + \sum x_j$. Hence *only one* l_{k1}^* can vanish (also for any given positive weights w_j).

With all $w_j = 1$ in (2.7) and (2.8), the condition $l_{k1}^* = 0$ becomes

$$\sum_{j=1}^{n} (z_k - z_j)\, u_j = 0, \qquad u_j = 1 \Big/ \prod_{i \neq j} | z_i - z_j |, \tag{3.1}$$

whence $z_k = \sum z_j u_j / \sum u_j$. As an average, z_k is in the relative interior of the convex hull of the z_j.

If $z_1 = 0$ and if for some $\epsilon \neq 1$, $|\epsilon| = 1$, $z_j \in E$ entails $z_j \epsilon \in E$, then by symmetry $\sum z_j u_j = 0$, hence $l_{11}^* = 0$.

When E has only 3 or 4 points, we have the following

THEOREM 1. *If $n = 3$, $l_{11}^* = 0$ if and only if z_1 lies between z_2 and z_3. For $n = 4$, $l_{11}^* = 0$ if and only if z_1 is the orthocenter of the acute-angled triangle (z_2, z_3, z_4).*

Proof. For $n = 3$, the result follows from (3.1) which reduces to $\operatorname{sgn}(z_1 - z_2) + \operatorname{sgn}(z_1 - z_3) = 0$.

For $n = 4$, (3.1) becomes

$$| z_3 - z_4 | \operatorname{sgn}(z_1 - z_2) + | z_4 - z_2 | \operatorname{sgn}(z_1 - z_3) + | z_2 - z_3 | \operatorname{sgn}(z_1 - z_4) = 0. \tag{3.2}$$

Equation (3.2) means that 3 vectors of lengths $| z_2 - z_3 |$, $| z_3 - z_4 |$, $| z_4 - z_2 |$, form a triangle. But the lengths of the sides of a triangle determine the angles except for factors ± 1. Hence but for a rotation there are only two possible positions:

$$| z_2 - z_3 | \operatorname{sgn}(z_2 - z_3) + | z_3 - z_4 | \operatorname{sgn}(z_3 - z_4) + | z_4 - z_2 | \operatorname{sgn}(z_4 - z_2) = 0 \tag{3.3}$$

or

$$\frac{| z_2 - z_3 |}{\operatorname{sgn}(z_2 - z_3)} + \frac{| z_3 - z_4 |}{\operatorname{sgn}(z_3 - z_4)} + \frac{| z_4 - z_2 |}{\operatorname{sgn}(z_4 - z_2)} = 0. \tag{3.4}$$

From these we see that $\operatorname{sgn}(z_1 - z_4)$, $\operatorname{sgn}(z_1 - z_2)$, $\operatorname{sgn}(z_1 - z_3)$ differ from $\operatorname{sgn}(z_2 - z_3)$, $\operatorname{sgn}(z_3 - z_4)$, $\operatorname{sgn}(z_4 - z_2)$ or their reciprocals only by a constant rotation factor. Therefore we have either

$$\frac{\operatorname{sgn}(z_1 - z_4)}{\operatorname{sgn}(z_2 - z_3)} = \frac{\operatorname{sgn}(z_1 - z_2)}{\operatorname{sgn}(z_3 - z_4)} = \frac{\operatorname{sgn}(z_1 - z_3)}{\operatorname{sgn}(z_4 - z_2)} \tag{3.5}$$

or

$$\begin{aligned}\operatorname{sgn}(z_1 - z_4) \operatorname{sgn}(z_2 - z_3) &= \operatorname{sgn}(z_1 - z_2) \operatorname{sgn}(z_3 - z_4) \\ &= \operatorname{sgn}(z_1 - z_3) \operatorname{sgn}(z_4 - z_2).\end{aligned} \tag{3.6}$$

Now (3.6) can not hold; for if it did then the three vectors $(z_1 - z_4)(z_2 - z_3)$, $(z_1 - z_2)(z_3 - z_4)$, $(z_1 - z_3)(z_4 - z_2)$ would have the same argument which is impossible since their sum is zero, and z_1, z_2, z_3, z_4 are distinct.

If z_2, z_3, z_4 are colinear and (3.5) holds, then (3.6) also holds. Hence z_2, z_3, z_4 are not colinear.

In case (3.5) holds, even if we allow, instead of equality, equality with $\pm$ factor, i.e.,

$$\frac{\operatorname{sgn}(z_1 - z_4)}{\operatorname{sgn}(z_2 - z_3)} = \pm \frac{\operatorname{sgn}(z_1 - z_2)}{\operatorname{sgn}(z_3 - z_4)} = \pm \frac{\operatorname{sgn}(z_1 - z_3)}{\operatorname{sgn}(z_4 - z_2)}, \tag{3.7}$$

it means that if we take lines through z_3, z_4, z_2 parallel to the lines $\overline{z_4z_2}$, $\overline{z_2z_3}$, $\overline{z_3z_4}$, respectively, and turn them about the same angle, then they should be concurrent at z_1. Now the point of intersection of any two of these lines while they are turning moves on a circle which is of the same size as the circumcircle of the triangle (z_2, z_3, z_4). These three circles meet at the orthocenter because the angle at the orthocenter and that at the vertex are supplementary. Thus the orthocenter is the only point z_1 fulfilling (3.7). But it is easy to see that (3.5) will be fulfilled if and only if z_1 is in the interior of the triangle (z_2, z_3, z_4). This completes the proof of the theorem for $n = 4$.

Remark 1. If $n = 4$ and the points $x_1 > x_2 > x_3 > x_4$ are real then the numbers $\operatorname{sgn} l_{1k}^*$, $\operatorname{sgn} l_{2k}^*$, $\operatorname{sgn} l_{3k}^*$, $\operatorname{sgn} l_{4k}^*$ are $1, -1, 1, -1$ for $k = 0$; $1, -1, -1, 1$ for $k = 1$; and $1, 1, -1, -1$ for $k = 2$.

Remark 2. The set of all sequences $(z_1, ..., z_n)$ for which $l_{11}^* = 0$ has dimension 5 for $n = 3$, but probably $2n - 2$ for $n > 3$. To prove it is $\geqslant 2n - 2$, let $z_1 = 0$, $z_j = \epsilon^{j-1}$ $(\epsilon^{n-1} = 1)$, $j > 1$. Then an arbitrary small variation of the z_j, $j > 1$, entails (if we want $l_{11}^* = 0$) a small variation of z_1 (for each j the Jacobian is $\neq 0$, so that the inverse function theorem can be applied).

We give now a second proof of the fact that the dimension mentioned above is $\geqslant 2n - 2$ for $n > 3$. From (3.1) we see that $l_{11}^* = 0$ if and only if

$$0 = \left[\operatorname{sgn}(z_2 - z_1) \Big/ \prod_{j \neq 1,2} | z_2 - z_j |\right] + \cdots.$$

Hence if $z_2, ..., z_n$ are real, $z_2 < \cdots < z_n$, and $y_k = 1/\prod_{j \neq k, j \neq 1} | z_k - z_j |$, then $l_{11}^* = 0$ will hold if and only if $z_k < z_1 < z_{k+1}$ where

$$y_2 + \cdots + y_k = y_{k+1} + \cdots + y_n. \tag{3.8}$$

Since for $n > 3$, $y_3 > y_2$, $y_{n-1} > y_n$, we have $3 \leqslant k \leqslant n - 2$. For $n = 3$ we get $z_2 < z_1 < z_3$; for $n = 4$ nonexistence of z_1; for $n \geqslant 5$, z_1 exists only for special positions of $z_2, ..., z_n$ (e.g., symmetry for odd n), and then z_1 can be chosen on an interval. For $n \geqslant 5$, any k, $3 \leqslant k \leqslant n - 2$, can occur;

indeed, for small $z_3 - z_2$ the left member is larger in the equation that is obtained from (3.8) by multiplying by the least common denominator, while for small $z_n - z_{n-1}$ the right member is larger, hence by continuity they can be equal. For $n = 5$, we have $k = 3$ and the condition (3.8) reduces to $z_3 - z_2 = z_5 - z_4$, i.e., symmetry.

To find a root of $F(z_1) = 0$ for general complex $z_2, \ldots, z_n$, where $F(z_1) = \sum_2^n y_k \operatorname{sgn}(z_k - z_1)$, note that for a fixed large $|z_1|$, $F(z_1)$ stays close to a circle about 0. If we contract the $|z_1|$-circle the image must at sometime pass through 0 unless 0 lies within, or on, one of the circles $F(z_k)$ with sgn 0 arbitrary of absolute value 1. (Note that for $k = 2, \ldots, n$, $F(z_k) = A_k + y_k e^{i\theta}$, $-\pi < \theta \leqslant \pi$, with $A_k = \sum_{\nu=2,\nu\neq k}^n y_\nu \operatorname{sgn}(z_\nu - z_k)$, is a circle with centre A_k and radius y_k). For example, for $n = 4$ this occurs (as seen by a simple computation) if and only if the triangle (z_2, z_3, z_4) has an angle ϕ, $\pi/2 \leqslant \phi \leqslant \pi$ at z_k. For $z_k = \epsilon^{k-1}$ ($\epsilon^{n-1} = 1$), 0 does not lie in, or on, these circles; for if, for instance, $0 = y_2\alpha + \sum_{k>2} y_k \operatorname{sgn}(1 - \epsilon^{k-2})$, $|\alpha| \leqslant 1$, then (since all y_k are equal)

$$\alpha = -\sum_1^{n-2} \operatorname{sgn}(1 - \epsilon^k) = -\cot \pi/(2n-2) < -1, \qquad n > 3.$$

Hence $F(z_1) = 0$ for some z_1; in fact, for $z_1 = 0$. For an arbitrary small change of the z_k, the circles still do not include 0, hence $F(z_1) = 0$ still has a root, which proves that the above-mentioned dimension is $\geqslant 2n - 2$.

4. Next-to-interpolatory Polynomials

Let Π_k denote the class of polynomials of degree $\leqslant k$, let $\tau(z)$ denote the next-to-interpolatory polynomial of degree $n - 2$ which minimizes the norm $\max_i\{w_i \,|f_i - t(z_i)|\}$ among all polynomials of degree $\leqslant n - 2$, where the w_i's are given positive constants. We now prove

Theorem 2. *The polynomials $L_{n1}(f; z)$ of (2.9) coincide with the next-to-interpolatory polynomials $\tau(z)$. Moreover if f is not a polynomial of degree $\leqslant n - 2$, then the following statements are equivalent:*

$$L_{n1}(f; z) = \tau(z) = f\Lambda\Delta'(1 \setminus k) \tag{4.1}$$

(in the notation of Sections 1 and 2), where $k = (1, 0, 0, \ldots)^T$ and b is given by $b_j^T = w_j^{-1}(\operatorname{sgn} j\Lambda k)^{-1}$.

$$\begin{aligned} w_i\,|f_i - L_{n1}(f; z_i)| &= |B_0|/\lambda_0^*, && \text{is independent of } i, \\ \arg \frac{f_i - L_{n1}(f; z_i)}{B_0} &= \arg \omega'(z_i), && \text{is independent of } f; \end{aligned} \tag{4.2}$$

$$L_{n1}(f; z) = \frac{1}{\lambda_0{}^*} \sum_1^n w_k^{-1} \Lambda_k(f; z) \mid l_{k0}^* \mid, \tag{4.3}$$

where $\Lambda_k(f; z)$ is the polynomial interpolating f in all points of E except z_k ; and $B_0 = \sum_1^n f_i l_{i0}^* \neq 0$.

Proof. We begin with the explicit formula for $\tau(z)$ given by Motzkin and Walsh [11]. They show that

$$B_0^{-1}\{L_{n0}(f; z) - \tau(z)\} = \omega(z) \sum_1^n \frac{\mu_i}{z - z_i}, \qquad \mu_i = w_i^{-1} \frac{l_{i0}^*}{\lambda_0{}^*}$$

Hence

$$\begin{aligned}\tau(z) &= \sum_1^n f_i l_{i0} - B_0 \omega(z) \sum_1^n \frac{\mu_i}{z - z_i} \\ &= \sum_1^n f_i l_{i0} - \frac{\lambda_0(z)}{\lambda_0{}^*} \sum_1^n f_i l_{i0}^* \\ &= \sum_1^n f_i l_{i1}(z),\end{aligned}$$

which proves the first part of (4.1). The second part is a reformulation of (2.9) in the notation of Section 1.

Equation (4.2) is a rewording of the equations which characterize $\tau(z)$ [11, p. 84]; corresponding results are given there for general families and weights. The second part of (4.2) can also be rewritten as

$$\arg \left\{ \frac{f_i - L_{n1}(f; z_i)}{\omega'(z_i)} \right\} = \arg B_0 \qquad \text{is independent of } i.$$

In this form the result (4.2) is equivalent to the conditions of Videnskii [3, 14]. In order to prove (4.3) we observe that from [10, Theorem 2] it follows (after a change of notation) that

$$\begin{aligned}\tau(z) &= \frac{1}{\lambda_0{}^*} \sum_1^n w_k^{-1} \Lambda_k(f) \mid l_{k0}^* \mid, \\ \Lambda_k(f) &= \sum_1^n f_j \lambda_{j,k}, \qquad \lambda_{j,k} = \prod_{h \neq j,k} \frac{z - z_h}{z_j - z_h}.\end{aligned} \tag{4.4}$$

We shall show that $\tau(z)$ given by (4.4) and $L_{n1}(f; z)$ given by (2.9) and (2.8) are equal. Now (4.4) implies that

$$\begin{aligned}\tau(z) &= \sum_{k=1}^n w_k^{-1} \mid l_{k0}^* \mid \sum_{j=1}^n f_j \lambda_{j,k} \Big/ \sum_{k=1}^n w_k^{-1} \mid l_{k0}^* \mid \\ &= \sum_{j=1}^n f_j \sum_{k=1}^n w_k^{-1} \mid l_{k0}^* \mid \lambda_{j,k} \Big/ \sum_1^n w_k^{-1} \mid l_{k0}^* \mid.\end{aligned}$$

It is therefore enough to show that $\lambda_0{}^* l_{j0} - l_{j0}^* \lambda_0 = \sum_1^n w_k^{-1} \mid l_{k0}^* \mid \lambda_{j,k}$ or equivalently that

$$\sum_{k=1}^{n} w_k^{-1}(l_{j0} - \lambda_{j,k}) \mid l_{k0}^* \mid = l_{j0}^* \sum_{1}^{n} w_k^{-1} l_{k0}/(\operatorname{sgn} l_{k0}^*) \tag{4.5}$$

since $\lambda_0{}^* = \sum_1^n w_k^{-1} \mid l_{k0}^* \mid$. We shall indeed show that

$$(l_{j0} - \lambda_{j,k})\, l_{k0}^* = l_{j0}^* l_{k0} \tag{4.6}$$

which implies (4.5). Now it is easy to see that

$$\lambda_{j,k} = ((z_j - z_k)/(z - z_k))\, l_{j0}\,,$$

so that $l_{j0} - \lambda_{j,k} = ((z - z_j)/(z - z_k))\, l_{j0}$. Since $l_{j0} = l_{j0}^* \omega(z)/(z - z_j)$ and $l_{k0} = l_{k0}^* \omega(z)/(z - z_k)$, we have proved (4.6). This completes the proof of Theorem 2.

5. Mean Square Convergence for Roots of Unity

Let $E = \{1, \alpha, \ldots, \alpha^{n-1}\}$ be the n roots of unity with $\alpha^n = 1$. Then $l_{j0} = (z^n - 1)/(z - \alpha^j) \cdot \alpha^j/n$, $j = 0, 1, \ldots, n-1$. Hence

$$\lambda_0(z) = \sum_{1}^{n} l_{j0}/(\operatorname{sgn} l_{j0}^*) = \frac{1}{n} \sum_{0}^{n-1} \frac{z^n - 1}{z - \alpha^j} = z^{n-1},$$

so that $\lambda_0{}^* = 1$. Then from (2.8) we have

$$l_{j1}(z) = l_{j0}(z) - \frac{l_{j0}^*}{\lambda_0{}^*} \lambda_0(z) = \frac{\alpha^{2j}}{n} \cdot \frac{z^{n-1} - \alpha^{j(n-1)}}{z - \alpha^j}\,.$$

Proceeding as before, we obtain, for $s = 0, 1, \ldots, n-1$,

$$L_{ns}(f; z) = (\alpha^{sj+j}/n) \cdot (z^{n-s} - \alpha^{j(n-s)})/(z - \alpha^j).$$

In this case because of the property $\sum_1^n \alpha^{jm} = 0$, $m = 1, \ldots, n-1$, we have

$$\begin{aligned} \sum_{j=0}^{n-1} \alpha^{jm} l_{js} &= z^m, \qquad m = 0, 1, \ldots, n-s-1, \\ &= 0, \qquad m = n-s, \ldots, n-1. \end{aligned} \tag{5.1}$$

We now state

Theorem 3. *Let $f(z)$ be analytic in D: $|z| < 1$, continuous in $D + C$,*

(C: $|z| = 1$). *Then the sequence of polynomials* $L_{nr}(f; z)$ *with* E *as the* n-th *roots of unity, converges to* $f(z)$ *in the* L_2-*norm. Consequently, for fixed* r,

$$\lim_{n\to\infty} L_{nr}(f; z) = f(z) \tag{5.2}$$

uniformly in $|z| \leqslant R < 1$.

Proof. Let $f(z) - t_{n-r-1}(z) = \delta(z)$, $e_n = \max[|\delta(z)|, z \in C]$, where $t_{n-r-1}(z)$ is the polynomial of degree $n - r - 1$ of best approximation to $f(z)$ on C. Then

$$\begin{aligned}\int_C |L_{nr}(f; z) - f(z)|^2 \,|dz| &\leqslant 2 \int_C |\delta(z)|^2 \,|dz| + 2 \int_C |L_{nr}(\delta(t); z)|^2 \,|dz \\ &\leqslant 2e_n^2 \cdot 2\pi + 2 \int_C |L_{nr}(\delta(t); z)|^2 \,|dz|.\end{aligned}$$

Since $\int_C z^m \bar{z}^n \,|dz| = 2\pi\delta_{nm}$, δ_{nm} being the Kronecker delta, we have

$$\int_C l_{jr}(z)\, l_{kr}(z) \,|dz| = \frac{2\pi}{n^2} \cdot \alpha^{(j-k)(r+1)} \sum_{h=0}^{n-r-1} \alpha^{h(j-k)}$$

so that

$$I_{k,j} = \left| \int_C l_{jr}(z)\overline{l_{kr}(z)} \,|dz| \right| \leqslant \begin{cases} \dfrac{2\pi}{n^2}(n - r), & \text{if } j = k, \\ \dfrac{2\pi}{n^2}(r + 1), & j \neq k. \end{cases}$$

Hence

$$\begin{aligned}\int_C |L_{nr}(\delta(t); z)|^2 \,|dz| &\leqslant \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} |\delta(\alpha^j)\, \overline{\delta(\alpha^k)}\, I_{k,j} \\ &\leqslant \frac{2\pi}{n^2} e_n^2 (n - r) + \frac{n(n-1)\, 2\pi(r+1)}{n^2} e_n^2 \\ &\leqslant 2\pi(r + 1)\, e_n^2.\end{aligned}$$

Since $e_n \to 0$ as $n \to \infty$, the theorem is proved.

To prove (5.2) one has only to observe that for $|z| \leqslant R < 1$,

$$L_{nr}(f; z) - f(z) = \frac{1}{2\pi i} \int_C \frac{L_{nr}(f; t) - f(t)}{t - z}\, dt.$$

Remark. For $r = 0$, Theorem 3 reduces to a theorem of Walsh and Sharma [16].

The following theorem is an analogue of a theorem of Fejér ([7], see also [12], p. 92) and is proved by the same method.

THEOREM 4. *If E denotes the set of n-th roots of* -1, *and* $L_{nr}(f; z)$ *the polynomials defined by the algorithm given by* (2.10)–(2.12), *then there exists a function* $f(z)$ *analytic in* $|z| < 1$ *and continuous in* $|z| \leqslant 1$ *for which*

$$\lim_{n\to\infty} L_{nr}(f; 1) = +\infty. \tag{5.3}$$

Proof. If $\beta_k = e^{(2k-1)\pi i/n}$, $k = 1,\ldots, n$, are the n-th roots of -1, we consider the polynomial

$$P_{2n}(z) = \frac{1}{n} + \frac{z}{n-1} + \cdots + \frac{z^{n-1}}{1} - \frac{z^{n+1}}{1} - \frac{z^{n+2}}{2} - \cdots - \frac{z^{2n}}{n}.$$

Then

$$P_{2n}(\beta_k) = \left(1 + \frac{1}{n-1}\right)\beta_k + \left(\frac{1}{2} + \frac{1}{n-2}\right)\beta_k^2 + \cdots + \left(\frac{1}{n-1} + 1\right)\beta_k^{n-1}$$

so that $L_{nr}(f; z) = \sum_1^n P_{2n}(\beta_k)\, l_{kr}(z)$, where

$$l_{kr}(z) = -\beta_k^{r+1}\,\frac{(z^{n-r} - \beta_k^{n-r})}{n(z - \beta_k)}.$$

Hence

$$L_{nr}(P_{2n}\,; z) = \left(1 + \frac{1}{n-1}\right) z + \left(\frac{1}{2} + \frac{1}{n-2}\right) z^2 + \cdots$$
$$+ \left(\frac{1}{n-r-1} + \frac{1}{r+1}\right) z^{n-r-1}$$

and

$$L_{nr}(P_{2n}\,; 1) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-r-1}\right)$$
$$+ \left(\frac{1}{r+1} + \cdots + \frac{1}{n-1}\right) > C\log n,$$

C being a fixed constant independent of n. Similarly, we can verify that if m is an odd integer,

$$L_{nr}(P_{2nm}\,; z) = \sum_{\nu=1}^{n-r-1} z^{\nu}\left[\left(\frac{1}{\nu} - \frac{1}{n+\nu} + \cdots + \frac{1}{n(m-1)+\nu}\right)\right.$$
$$\left. + \left(\frac{1}{n-\nu} - \frac{1}{2n-\nu} + \cdots + \frac{1}{nm-\nu}\right)\right].$$

Then for $2m < n - r$,

$$L_{nr}(P_{2m}\,; 1) = P_{2m}(1) = 0,$$

and for $m \geqslant 3$,

$$L_{nr}(P_{2nm}\,;1) = \sum_{\nu=1}^{n-r-1} + \sum_{\nu=r+1}^{n-1} \left\{\frac{1}{\nu} - \frac{1}{n+\nu} + \cdots + \frac{1}{n(m-1)+\nu}\right\} > 0.$$

Set

$$f(z) = \sum_{s=1}^{\infty} \frac{P_{2\cdot 3^{s^3}}(z)}{s^2}.$$

Since $|P_{2n}(e^{i\theta})| \leqslant \int_0^\pi \sin\theta/\theta \, d\theta = 2\lambda$, $f(z)$ is analytic in $|z| < 1$ and continuous in $|z| \leqslant 1$. However,

$$L_{3^{n^3},r}(f;1) = \sum_{s=1}^{\infty} L_{3^{n^3},r}(P_{2\cdot 3^{s^3}}(z);1)/s^2 > L_{3^{n^3},r}(P_{2\cdot 3^{n^3}}(z);1)/n^2 = Cn$$

so that $\overline{\lim}\, L_{nr}(f;1) = \infty$, which completes the proof of the theorem.

6. Relation with Taylor's Expansion

The following theorem establishes a close connection between the polynomials $L_{nr}(f;z)$ based on the roots of unity and the Taylor expansion of $f(z)$ about the origin. For $r = 0$, this theorem is due to Walsh [15, p. 153].

Theorem 5. *If $f(z)$ is analytic in $|z| < \rho$ $(\rho > 1)$ and if $P_{n-r-1}(z)$ is the polynomial of degree $n-r-1$ taken from the Taylor expansion of $f(z)$ about the origin then $L_{nr}(f;z) - P_{n-r-1}(z) \to 0$ uniformly in $|z| \leqslant R < \rho^2$ as $n \to \infty$.*

Proof. We shall need the following representation for $L_{nr}(f;z)$.

$$L_{nr}(f;z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,\frac{t^r(t^{n-r}-z^{n-r})}{t^n-1}\,dt, \tag{6.1}$$

where C is the circle $|z| = R$, $1 < R < \rho$. Since

$$f_j = f(\alpha^j) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-\alpha^j}\,dt,$$

we have

$$L_{nr}(f;z) = \frac{1}{2\pi i}\int_C f(t)\cdot\frac{1}{n}\sum_{j=0}^{n-1}\frac{\alpha^{jr+j}}{t-\alpha^j}\cdot\frac{z^{n-r}-\alpha^{j(n-r)}}{z-\alpha^j}\,dt. \tag{6.2}$$

Using the identities

$$1/(t-\alpha^j)(z-\alpha^j) = (1/t-z)[(1/z-\alpha^j)-(1/t-\alpha^j)],$$

$$\frac{1}{n}\sum_{0}^{n-1}\frac{\alpha^{mj}}{z-\alpha^j} = \frac{z^{m-1}}{z^m-1}, \qquad m = 1,..., n,$$

we can show that for $m = 1, 2,..., n$ we have

$$\frac{1}{n}\sum_{j=0}^{n-1}\frac{\alpha^{jm}}{(z-\alpha^j)(t-\alpha^j)} = \left(\frac{z^{m-1}}{z^m-1} - \frac{t^{m-1}}{t^m-1}\,\frac{1}{t-z}\right). \tag{6.3}$$

Combining (6.2) and (6.3) we have (6.1). Since

$$f(z) - P_{n-r-1}(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\cdot\left(\frac{z}{t}\right)^{n-r} dt,$$

we have from (6.1)

$$P_{n-r-1}(z) - L_{nr}(f; z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\cdot\frac{z^{n-r}-t^{n-r}}{(t^n-1)\,t^{n-r}}\,dt.$$

If $|z| = Z$, then the right side tends uniformly to zero as $(R^{n-r}+Z^{n-r})/R^{n-r}(R^n-1)$ approaches zero which occurs if $Z < R^2$. This completes the proof of the theorem.

If $f(z) = (z-\rho)^{-1}$, then it is easy to verify that

$$f(z) - L_{nr}(f; z) = (z^{n-r}\rho^r - 1)/(z-\rho)(\rho^n-1).$$

Also

$$f(z) - P_{n-r-1}(z) = z^{n-r}/\rho^{n-r}(z-\rho)$$

so that

$$L_{nr}(f; z) - P_{n-r-1}(z) = (\rho^{n-r} - z^{n-r})/\rho^{n-r}(z-\rho)(\rho^n-1).$$

For $z = \rho^2$,

$$L_{nr}(f; z) - P_{n-r-1}(z) = (1-\rho^{n-r})/(\rho^2-\rho)(\rho^n-1)$$

which tends to $\rho^{-r-1}(1-\rho)^{-1}$ as $n \to \infty$. This shows that the result is the best possible.

7. Maximal convergence for Fekete Points

If K is connected and regular (see [15, p. 170]), then K possesses a Green's function $G(x, y)$ with pole at infinity. In fact the function $\omega = \phi(z) = e^{G+iH}$, where H is conjugate to G in K, maps K conformally onto the exterior of the unit circle γ in the ω-plane so that points at infinity correspond to each other. C_ρ will indicate the locus $G(x, y) = \log \rho > 0$, or $|\phi(z)| = \rho > 1$.

We now establish

THEOREM 6. *Let C be a closed bounded point set whose complement K is connected and regular. Let $E = \{z_1^{(n)},..., z_n^{(n)}\}$ be a set of n points which maximizes $|V_n(z_1,..., z_n)|$ for points $z_1,..., z_n$ on C, V_n being the familiar Vandermonde determinant. If $f(z)$ is single-valued and analytic on C, then $L_{nr}(f; z)$ converges maximally to $f(z)$ on C.*

For $r = 0$, the result is due to Fekete [15, p. 170].

Proof. Let ρ be a number >1 such that $f(z)$ is single-valued and analytic inside C_ρ. Let R be given, $1 < R < \rho$. Then there exist polynomials $\pi_{n-r-1}(z)$ of degree $n - r - 1$ such that

$$|f(z) - \pi_{n-r-1}(z)| \leqslant M/R^n, \qquad z \in C. \tag{7.1}$$

Hence for $z \in C$,

$$\begin{aligned} | L_{nr}(f; z) - f(z)| &\leqslant |f(z) - \pi_{n-r-1}(z)| + | L_{nr}(f - \pi_{n-r-1}; z)| \\ &\leqslant \frac{M}{R^n} + \frac{2^r M}{R^n} \sum_1^n | l_{k0}(z)|, \end{aligned}$$

where the last inequality follows from (7.1) and Lemma 2.

Since by the definition of $\{z_k^{(n)}\}_1^n$ we have

$$| l_{k0}(z)| = | \omega(z)/(z - z_\nu^{(n)}) \, \omega'(z_\nu^{(n)})| \leqslant 1,$$

it follows that $| L_{nr}(f; z) - f(z)| \leqslant (M/R^n)(1 + n \cdot 2^r)$, so that

$$\overline{\lim_{n\to\infty}} \, [\max | f(z) - L_{nr}(f; z)|, \, z \text{ on } C]^{1/n} \leqslant \frac{1}{R},$$

which proves the theorem.

8. Real Abscissas (Mean Square Convergence)

We consider now the case where E is a set of n real points $x_1, x_2,..., x_n$ lying in $[-1, 1]$ and forming the n-th row of a triangular matrix E. To be precise we should indicate these by $x_1^{(n)},..., x_n^{(n)}$, but for the sake of simplicity, we avoid the superscripts. Let $w(x) \geqslant 0$ be a given weight function on $[-1, 1]$ with $\int_{-1}^1 w(x)\, dx = 1$ and let $\{Q_n(x)\}_0^\infty$ denote the sequence of n-th degree orthonormal polynomials on $[-1, 1]$ with respect to the weight function $w(x)$. We shall make the following hypothesis (H) about $x_1, x_2, \ldots, x_n$:

(H) $x_1, x_2, \ldots, x_n$ are the zeros of the polynomial

$$\omega(x) = Q_n(x) + A_n Q_{n-1}(x) \tag{8.1}$$

where A_n is a constant such that the zeros of $\omega(x)$ are real and distinct and lie in $[-1, 1]$.

We have

THEOREM 7. *Let the nodes $\{x_i\}_1^n$ satisfy (H). If $f(x)$ is continuous on $[-1, 1]$, then for any fixed integer $r \geqslant 0$, the polynomials $L_{nr}(f; x)$ have the property*

$$\lim_{n\to\infty} \int_{-1}^{1} \{L_{nr}(f; x) - f(x)\}^2 w(x)\, dx = 0. \tag{8.2}$$

If $w(x) \geqslant M > 0$, we have

$$\lim_{n\to\infty} \int_{-1}^{1} \{L_{nr}(f; x) - f(x)\}^2\, dx = 0. \tag{8.3}$$

Proof. We shall prove (8.2) from which (8.3) follows at once. Let $R(x)$ be the polynomial which best approximates $f(x)$ on $[-1, 1]$ in the uniform norm among all polynomials of degree $n - r - 1$ and let $\max_x |f(x) - R(x)| = e_n$. Then $e_n \to 0$ as $n \to \infty$. Setting $g(t) = f(t) - R(t)$ and keeping in mind the linearity of the operator L_{nr} and its reproducing property (Lemma 1), i.e., $L_{nr}(R; x) = R(x)$, we have

$$\begin{aligned}\int_{-1}^{1} &\{L_{nr}(f; x) - f(x)\}^2 w(x)\, dx \\ &\leqslant 2 \int_{-1}^{1} \{L_{nr}(f; x) - R(x)\}^2 w(x)\, dx + 2 \int_{-1}^{1} (f(x) - R(x))^2 w(x)\, dx \\ &\leqslant 2e_n^2 + 2 \int_{-1}^{1} (L_{nr}(g; x))^2 w(x)\, dx.\end{aligned} \tag{8.4}$$

Since the fundamental polynomials of Lagrange interpolation $l_{k0}(x)$ have the orthogonality property:

$$\int_{-1}^{1} l_{j0}(x)\, l_{k0}(x)\, w(x)\, dx = 0, \qquad j \neq k,$$

we have on using (2.17):

$$\begin{aligned}\int_{-1}^{1} (L_{nr}(g; x))^2\, w(x)\, dx &= \int_{-1}^{1} \left(\sum_1^n \alpha_{k,r} l_{k0}\right)^2 w(x)\, dx \Big/ \prod_0^{r-1} (\lambda_k^*)^2 \\ &= \sum_1^n (\alpha_{k,r})^2 \int_{-1}^{1} l_{k0}^2(x)\, w(x)\, dx \Big/ \prod_0^{r-1} (\lambda_k^*)^2,\end{aligned} \tag{8.5}$$

where $|\alpha_{k,r}| \leqslant 2^r e_n \cdot \prod_0^{r-1} \lambda_k^*$. Now $l_{k0}^2 - l_{k0}$ vanishes for $x_1, ..., x_n$ so that $l_{k0}^2 - l_{k0} = \omega(x)\, S_{n-2}(x)$ whence, from the orthogonality of the Q_i's, we have

$$\int_{-1}^{1} l_{k0}^2 w(x)\, dx = \int_{-1}^{1} l_{k0} w(x)\, dx.$$

Hence from (8.5) we have

$$\int_{-1}^{1} (L_{nr}(g; x))^2\, w(x)\, dx \leqslant 2^{2r} \cdot e_n^2 \int_{-1}^{1} \sum_1^n l_{k0}(x) \cdot w(x)\, dx = 2^{2r} \cdot e_n^2$$

so that (8.4) yields

$$\int_{-1}^{1} (L_{nr}(f; x) - f(x))^2\, w(x)\, dx \leqslant (2^{2r+1} + 2)\, e_n^2$$

which proves (8.2).

Remark 1. Theorem 7 holds even when the nodes $x_1, ..., x_n$ satisfy a more general condition, namely, that they be the zeros of the polynomials $\omega(x) = Q_n + A_n Q_{n-1} + B_n Q_{n-2}$, $B_n \leqslant 0$, where A_n, B_n are real constants such that the zeros of $\omega(x)$ are real, distinct and lie in $[-1, 1]$. Also the function f may be taken to be only R-integrable. The proof of Theorem 7 can be modified as in Erdös–Turán [4] to yield the stronger version.

Remark 2. We have, *a fortiori*, for $w(x) \geqslant M > 0$,

$$\lim_{n\to\infty} \int_{-1}^{1} |f(x) - L_{nr}(f; x)|\, dx = 0.$$

9. Strong Mean Convergence

We shall show that if $x_1, ..., x_n$ are the zeros of the Tchebycheff polynomial $T_n(x) = \cos(n \arc\cos x)$, then a result stronger than Theorem 7 holds. More precisely, we shall prove

THEOREM 8. *If the nodes* $\{x_i\}_1^n$ *are the zeros of* $T_n(x)$ *and if* $f(x)$ *is continuous in* $[-1, 1]$, *then*

$$\lim_{n\to\infty} \int_{-1}^{1} [L_{nr}(f; x) - f(x)]^4 \, dx = 0. \tag{9.1}$$

Proof. Since

$$0 < \int_{-1}^{1} \{L_{nr}(f; x) - f(x)\}^4 \, dx \leqslant \int_{-1}^{1} \{L_{nr}(f; x) - f(x)\}^4 \frac{dx}{\sqrt{1-x^2}},$$

it is enough to prove that

$$\lim_{n\to\infty} \int_{-1}^{1} \{L_{nr}(f; x) - f(x)\}^4 \frac{dx}{\sqrt{1-x^2}} = 0. \tag{9.2}$$

Proceeding as in the proof of Theorem 7, we may use the polynomial $R(x)$ of degree $n - r - 1$ of best approximation to $f(x)$ on $[-1, 1]$ and $e_n = \max_x |f(x) - R(x)|$. It is easy to see that in order to prove (9.2), it is sufficient to show that

$$\int_0^{\pi} \{L_{nr}(g(t); \theta)\}^4 \, d\theta \equiv \int_{-1}^{1} \{L_{nr}(g(t); x)\}^4 \frac{dx}{\sqrt{1-x^2}}$$

is bounded as $n \to \infty$. From (2.17) we see that $L_{nr}(f; x) = L_{n0}(\Delta; x)$ where $\Delta(x_k) = \alpha_{k,r} \prod_0^{r-1} \lambda_j^*$, $k = 1,..., n$. Then the result of Feldheim [7, p. 30] applies and we have

$$\int_0^{\pi} \{L_{n0}(g(t); \theta)\}^4 \, d\theta \leqslant (C_1 + C_2 + 2\pi)\, 2^{4r} \cdot e_n^{\,4},$$

which completes the proof of (9.2).

It follows by using the reasoning of Erdös and Feldheim [4] that if $\{x_i\}_1^n$ are the zeros of $T_n(x)$ and if $f(x)$ is continuous in $[-1, 1]$, then the following stronger result holds:

$$\lim_{n\to\infty} \int_{-1}^{1} | L_{nr}(f; x) - f(x)|^p \, dx = 0, \qquad p = 1, 2, 3,\ldots . \tag{9.3}$$

Following Feldheim [8] we shall also prove

THEOREM 9. *If* $\{x_i\}_1^n$ *are the zeros of* $U_n(x)$ (*the Tchebycheff polynomials of second kind*) *then there exists a function* $f(x)$ *continuous in* $[-1, 1]$ *such that*

$$\lim_{n\to\infty} \int_{-1}^{1} \{L_{nr}(f; x) - f(x)\}^2 \, dx = +\infty. \tag{9.4}$$

For $r = 0$, this result is due to Feldheim [9, p. 77].

Proof. We begin with the identity

$$\sum_{\nu=1}^{n} (-1)^{\nu-1} U_r(x_\nu)\, l_{\nu r}(x) \equiv U_{n-r-1}(x), \tag{9.5}$$

which follows from the observations that

$$U_{n-r-1}(x_\nu) = \sin \frac{(n-r)\nu\pi}{n+1} \Big/ \sin \frac{\nu\pi}{n+1} = (-1)^{\nu+1} U_r(x_\nu)$$

and $L_{nr}(U_{n-r-1}\,; x) = U_{n-r-1}(x)$. For $r = 0$, (9.5) is the known identity

$$\sum_{1}^{n} (-1)^{\nu+1} l_{\nu 0}(x) \equiv U_{n-1}(x).$$

Since $U_{n-r-1}^2(x) = \sum_0^{n-r-1} U_{2k}(x)$, we have from (9.5):

$$\sum_{i=1}^{n} \sum_{k=1}^{n} (-1)^{i+k} U_r(x_i)\, U_r(x_k) \int_{-1}^{1} l_{ir}(x)\, l_{kr}(x)\, dx$$

$$= \int_{-1}^{1} U_{n-r-1}^2(x)\, dx = \sum_{\nu=0}^{n-r-1} \int_{-1}^{1} U_{2\nu}(x)\, dx$$

$$= \sum_{\nu=0}^{n-r-1} \frac{2}{2\nu+1} > \log \frac{2(n-r)}{3}.$$

Consider the function $f_n(x)$ which is piecewise linear between the x_j's and satisfies $f_n(x_\nu) = (-1)^\nu U_r(x_\nu)/(r+1)$, $\nu = 1,\ldots, n$. For $x \geqslant x_n$ and $x \leqslant x_1$ let $f_n(x)$ be constant. Then $|f_n(x)| \leqslant 1$. Also

$$\int_{-1}^{1} (L_{nr}(f_n\,; x))^2\, dx > \log \frac{2(n-r)}{3}.$$

By the Weierstrass approximation theorem there exists a polynomial $\phi_m(x)$, of degree $m = m(n)$, such that

$$|\phi_m(x)| \leqslant \frac{3}{2}, \qquad |x| = 1,$$

$$\int_{-1}^{1} (L_{nr}(\phi_m\,; x))^2\, dx > \frac{1}{2} \log \frac{2(n-r)}{3}, \qquad n = r+1, \quad r+2,\ldots .$$

Set $f(x) = \sum_{\nu=1}^{n} C_\nu \phi_{n_\nu}(x)$ where $C_1 = n_1 = r+1$ and where the coefficients C_ν and the indices n_ν are determined as follows:

$$C_{k+1} = \min \left\{ \frac{C_k}{4}, \frac{1}{\max_{|x|\leqslant 1} \sum_{\nu=1}^{n_k} |l_{\nu r}^{(n_k)}(x)|} \right\}, \qquad k = 1, 2,\ldots$$

and n_{k+1} is the smallest integer for which $n_{k+1} > m(n_k) + 1$. Then it can be shown, exactly as in [9] and in the earlier paper [5] that $f(x)$ is continuous and that (9.4) holds.

10. Concluding Remarks

10.1. By the method of Turán [13] we can show that if $\{x_i\}_1^n$ are the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, and if $f \in C[-1, 1]$, then

$$\lim_{n\to\infty} \int_{-1}^{1} (f(x) - L_{nr}(f; x))^2 \, dx = 0$$

if $\max(\alpha, \beta) < 1/2$, and

$$\lim_{n\to\infty} \int_{-1}^{1} | f(x) - L_{nr}(f; x)| \, dx = 0$$

if $\max(\alpha, \beta) < 3/2$.

Following the reasoning of Askey [1] it can be proved for the same $\{x_i\}_1^n$ and $\alpha = \beta \geqslant 1/2$ that

$$\lim_{n\to\infty} \int_{-1}^{1} | L_{nr}(f; x) - f(x)|^p (1 - x^2)^\alpha \, dx = 0 \tag{10.1}$$

if $p < 4(\alpha + 1)/(2\alpha + 1)$, and that if $p \geqslant 4(\alpha + 1)/(2\alpha + 1)$, there exists a continuous function $f(x)$ for which (10.1) fails.

10.2. It is easy to prove a generalization of a result of Fejér [6]: if the Lebesgue constant $\lambda_n(E) = \max_x \sum_1^n | l_{k0}(x)| < c_1 n^\beta$, $0 < \beta < 1$, then $L_{nr}(f; x)$ converges uniformly to $f(x)$ in $[-1, 1]$ if $f \in \text{Lip}\, \gamma$, $\gamma > \beta$. Indeed if $Q(x)$ is the polynomial of degree $n - r - 1$ approximating best to $f(x)$ in $[-1, 1]$ in the uniform norm, then

$$| f(x) - Q(x)| \leqslant c_2 n^{-\gamma}.$$

Using the reproducing property of $L_{nr}(f, x)$ we have, by Lemma 2,

$$\begin{aligned} | L_{nr}(f, x) - f(x)| &\leqslant | L_{nr}(f - Q; x)| + | Q(x) - f(x)| \\ &\leqslant c_2 n^{-\gamma} + 2^r \cdot c_2 n^{-\gamma} \cdot \sum_1^n | l_{k0}(x)| \\ &\leqslant c_2 n^{-\gamma} + c_3 n^{\beta-\gamma}; \end{aligned}$$

the assertion follows because $\gamma > \beta$.

10.3. Using the method of Curtis [2] for L_{n0} and L_{n1} we see, because of Lemma 1, that for every given matrix E there exists a continuous function $f \in C[-1, 1]$ such that $L_{nr}(f, x)$ fails to converge uniformly in $[-1, 1]$.

10.4. We have not been able to prove the analog of Bernstein's result which asserts that for $f_0(x) \equiv |x|$ and for equidistant abscissas, $L_{n0}(f_0\,; x)$ converge to f_0 at no point of $[-1, 1]$ except $(-1, 0, 1)$. It would be interesting to find sets of nodes for which the operator sequence $L_{nr}(f, x)$ converges to $f(x)$ for fixed $r \geqslant 1$ in some norm while $L_{n0}(f, x)$ does not. [The converse cannot occur, because of (2.15).]

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